Jointly Modelling the Body and Tail of Bivariate Data

Lídia André

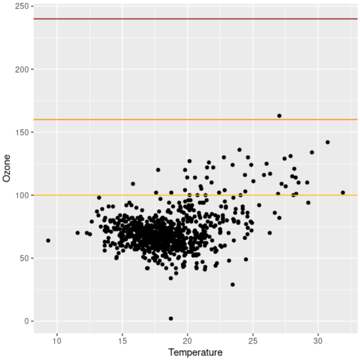
Supervisors: Jennifer Wadsworth and Adrian O'Hagan

November 21st, 2022

Engineering and Physical Sciences Research Council

Problem

Sometimes we may be interested in not only modelling the extremes but also the bulk of the distribution accurately - *e.g.* environmental applications



Univariate Framework

- There have been proposed parametric, semi-parametric and non-parametric models

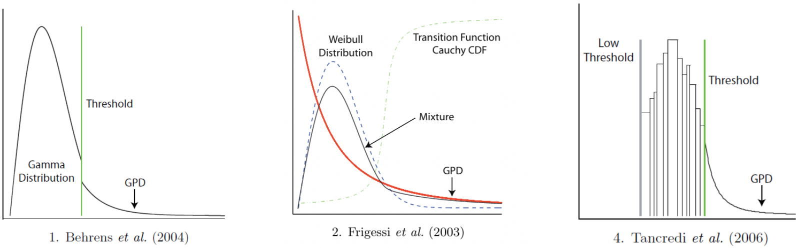


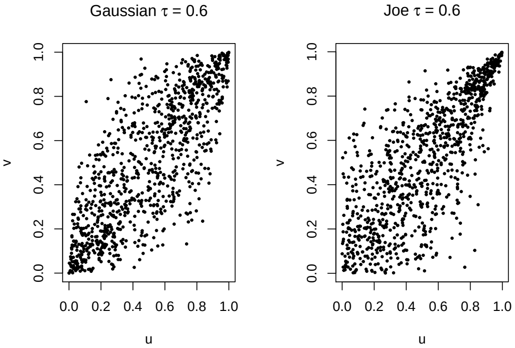
Figure 1: Taken from Scarrott and MacDonald (2012)

Introduction to Copulas

In a multivariate setting we are also concerned about the dependence between variables. A way of measuring it is by using **copulas**

A copula C is a **joint distribution** of a random vector $(X_1, \ldots, X_d)$

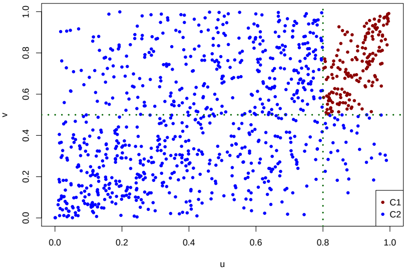
$$F(x_1, \ldots, x_d) = C\left(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\right), \quad d \geq 2$$



Multivariate Framework

Aulbach et al. (2012) model the whole data set by fitting one copula to the body and another to the upper tail

- It sometimes doesn't offer a smooth transition between the two copulas
- It requires the choice of thresholds
- The likelihood of the model doesn't have a closed form so no inference was done



Weighted Copula Model

For $(u^*, v^*) \in [0,1]^2$, we define the density c^* as

$$c^*(u^*, v^*; \gamma) = \frac{\pi(u^*, v^*; \theta) c_t(u^*, v^*; \boldsymbol{\alpha}) + [1 - \pi(u^*, v^*; \theta)] c_b(u^*, v^*; \boldsymbol{\beta})}{K(\gamma)}$$

[1] For more details see André et al. (2022)

Weighted Copula Model

For $(u^*, v^*) \in [0,1]^2$, we define the density c^* as

$$c^*(u^*, v^*; \gamma) = \frac{\pi(u^*, v^*; \theta) c_t(u^*, v^*; \boldsymbol{\alpha}) + [1 - \pi(u^*, v^*; \theta)] c_b(u^*, v^*; \boldsymbol{\beta})}{K(\gamma)}$$

- $c_t, \; c_b \rightarrow$ copula densities tailored to the tail and body, respectively.

[1] For more details see André et al. (2022)

Weighted Copula Model

For $(u^*, v^*) \in [0,1]^2$, we define the density c^* as

$$c^*(u^*, v^*; \boldsymbol{\gamma}) = \frac{\pi(u^*, v^*; \theta) c_t(u^*, v^*; \boldsymbol{\alpha}) + [1 - \pi(u^*, v^*; \theta)] c_b(u^*, v^*; \boldsymbol{\beta})}{K(\boldsymbol{\gamma})}$$

- $c_t, \; c_b \rightarrow$ copula densities tailored to the tail and body, respectively.
- $\pi(u^*, v^*; \theta) \rightarrow$ dynamic weighting function, defined in $[0,1]^2$ and increasing in u^* and v^*

[1]For more details see André et al. (2022)

Weighted Copula Model

For $(u^*, v^*) \in [0,1]^2$, we define the density c^* as

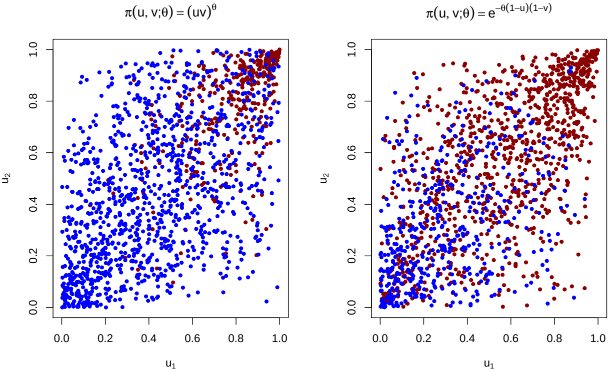
$$c^*(u^*, v^*; \boldsymbol{\gamma}) = \frac{\pi(u^*, v^*; \theta) c_t(u^*, v^*; \boldsymbol{\alpha}) + [1 - \pi(u^*, v^*; \theta)] c_b(u^*, v^*; \boldsymbol{\beta})}{K(\boldsymbol{\gamma})}$$

- $c_t, \; c_b \rightarrow$ copula densities tailored to the tail and body, respectively.
- $\pi(u^*, v^*; \theta) \rightarrow$ dynamic weighting function, defined in $[0,1]^2$ and increasing in u^* and v^*
- $\boldsymbol{\gamma} = (\theta, \boldsymbol{\alpha}, \boldsymbol{\beta}) \rightarrow$ vector of model parameters
- $K(\boldsymbol{\gamma}) \rightarrow$ normalising constant [1]

[1] For more details see André et al. (2022)

Weighted Copula Model

- Doesn't require a choice of threshold
- Offers a smooth transition between the body and tail copulas
- However, it is also hard to perform inference on it



Inference

The inference on the model was achieved by fitting the copula of the density c^* via numerical integration as follows

$$c(u, v; \gamma) = \frac{c^*\left(F_{U^*}^{-1}(u), F_{V^*}^{-1}(v); \gamma\right)}{f_{U^*}\left(F_{U^*}^{-1}(u)\right) f_{V^*}\left(F_{V^*}^{-1}(v)\right)}$$

where

$$F_{U^*}(u^*) = P[U^* \leq u^*] = \int_0^{u^*} \int_0^1 c^*(u, v) \mathrm{d}v \mathrm{d}u$$
$$f_{U^*}(u^*) = \int_0^1 c^*(u^*, v)\, \mathrm{d}v, \qquad v \in (0, 1)$$

Extremal Dependence Properties

It is important to know if extreme values of the variables are likely to occur together (**asymptotic dependence**) or not (**asymptotic independence**)

$$\chi = \lim_{r \to 1} P\left[F_Y(y) > r \mid F_X(x) > r\right],$$

$$P\left[F_Y(y) > r \mid F_X(x) > r\right] \sim \mathcal{L}(1-r)(1-r)^{\frac{1}{\eta}-1} \qquad \text{as } r \to 1$$

- Asymptotic Dependence: $\chi > 0$ and $\eta = 1$
- Asymptotic Independence: $\chi = 0$ and $\eta \neq 1$

Extremal Dependence Properties

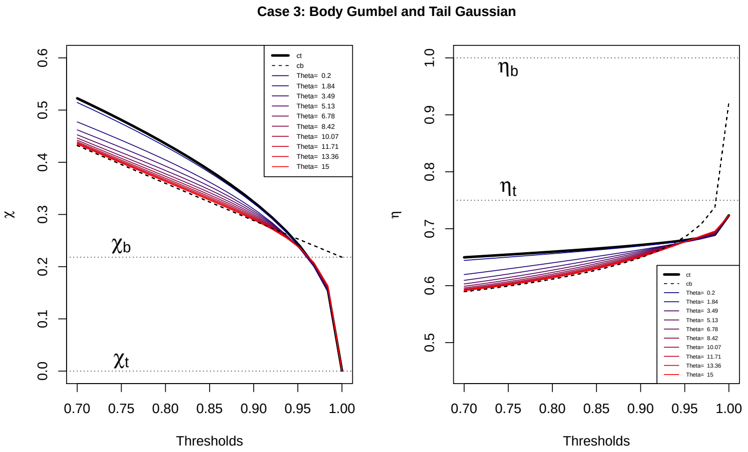


Figure 2: Weight function: $\pi(u^*, v^*; \theta) = (u^* v^*)^{\theta}$.

Extremal Dependence Properties

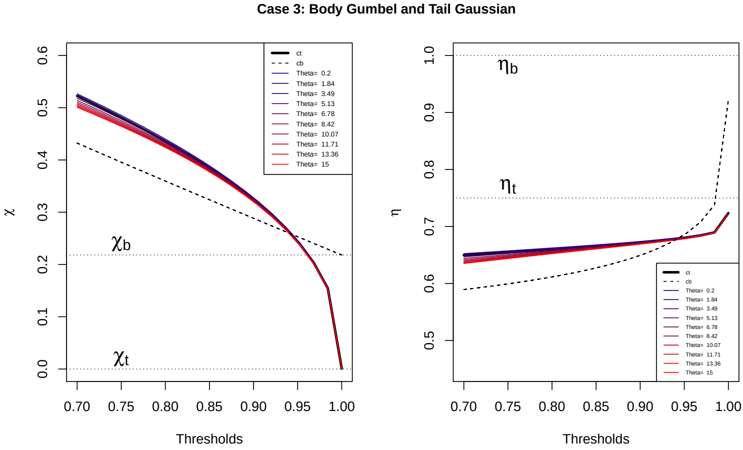


Figure 3: Weight function: $\pi(u^*, v^*; \theta) = \exp\{-\theta(1 - u^*)(1 - v^*)\}$.

Ozone and Temperature Data

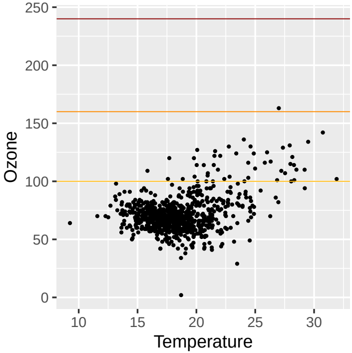
- Temperature may influence the levels of Ozone concentration in the air
- The legal thresholds for O_3 levels in the UK might then be found in the body and not just in the tails of the data

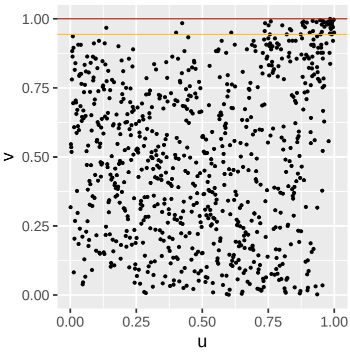
UK legal thresholds:

Levels	Low	Moderate	High	Very High
$O_3\,(\mu g/m^3)$	$[0, 100]$	$[101, 160]$	$[161, 240]$	> 240

We applied our model to the summers between 2011 and 2019 of Blackpool, UK

Ozone and Temperature Data



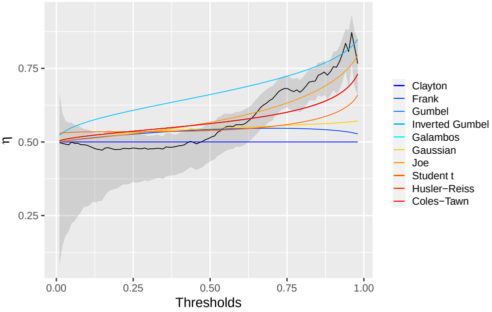


Apart from the upper tail, the variables seem to be negative correlated

Ozone and Temperature Data

Fitting a single copula

Copula	AIC
Clayton	2.0
Gaussian	-28.6
Frank	-15.8
Joe	**-143.6**
Gumbel	-97.4
Student t	-52.8
Inverted Gumbel	0.1
Hüsler-Reiss	-99.1
Coles-Tawn	-99.0
Galambos	-95.9



None of the single copulas showed negative correlation

Ozone and Temperature Data

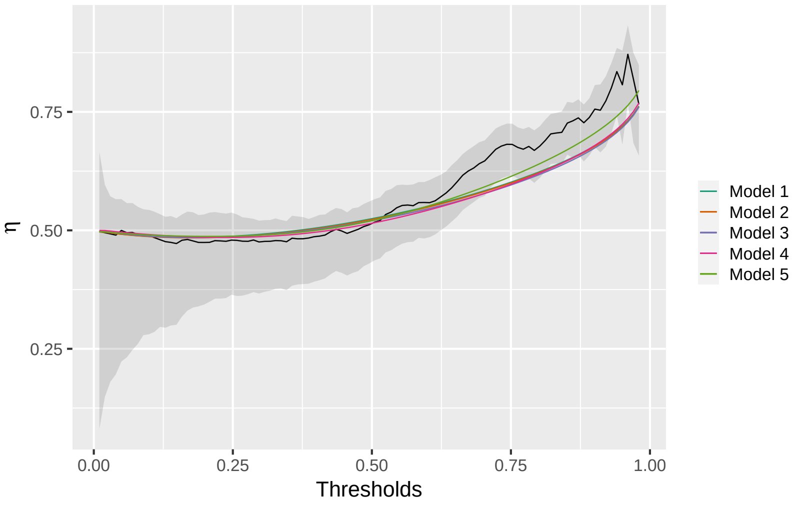
Fitting the weighted copula model with

$$\pi(u^*, v^*; \theta) = (u^* v^*)^\theta$$

Model		Parameters			AIC
c_b	c_t	$\hat{\beta}$	$\hat{\boldsymbol{\alpha}}$	$\hat{\theta}$	
Gaussian	Hüsler-Reiss	-0.40	1.24	0.35	-176.1
Gaussian	Galambos	-0.41	0.79	0.34	-172.1
Gaussian	Coles-Tawn	-0.33	0.35, 2.86	0.43	-158.4
Frank	Coles-Tawn	-2.52	0.33, 4.80	0.37	-163.2
Frank	Joe	-4.11	1.61	0.18	**-184.9**

The models with the best AIC all show negative correlation in the copulas tailored to the body

Ozone and Temperature Data



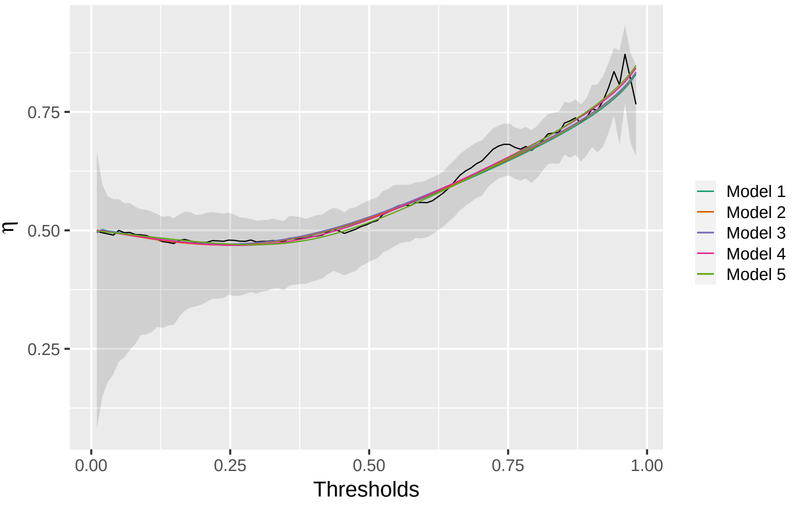
Ozone and Temperature Data

Fitting the weighted copula model with

$$\pi(u^*, v^*; \theta) = \exp\{-\theta(1-u^*)(1-v^*)\}$$

Model		Parameters			AIC
c_b	c_t	$\hat{\beta}$	$\hat{\boldsymbol{\alpha}}$	$\hat{\theta}$	
Gaussian	Hüsler-Reiss	-0.74	1.33	3.32	**-240.1**
Gaussian	Galambos	-0.72	0.90	3.55	-237.2
Gaussian	Coles-Tawn	-0.74	0.85, 0.79	3.25	-234.8
Frank	Coles-Tawn	-4.51	0.87, 1.02	4.33	-235.7
Frank	Joe	-6.49	1.72	2.45	-232.9

Ozone and Temperature Data



Ozone and Temperature Data

Other diagnostics

Models	Kendall's τ	$P[T \geq 24, O_3 \geq 100]$	$P[O_3 \geq 100 \mid 22 \leq T \leq 23]$
Empirical	0.0812	0.0302	0.1330
(95% CI)	(0.0173 , 0.1867)	(0.0147 , 0.0544)	(0.0227 , 0.1944)
Model 1	0.0690	0.0246	0.1441
Model 2	0.0663	0.0250	0.1412
Model 3	0.0770	0.0251	0.1429
Model 4	0.0779	0.0262	0.1392
Model 5	0.0718	0.0267	0.1366

Conclusions

- Our model provides a better fit than just fitting a single copula to the data
- It is flexible - it is able to capture different structures within the same data set
- However, it is computationally expensive
- Further Steps:
    - Account for non-stationarity - incorporate covariates

Questions?

Thank you all for listening!

References I

André, L. M., Wadsworth, J. L., and O'Hagan, A. (2022). Joint modelling of the body and tail of bivariate data *(preprint)*.

Aulbach, S., Bayer, V., and Falk, M. (2012). A Multivariate Piecing-Together Approach with an Application to Operational Loss Data. *Bernoulli*, 18:455–475.

Scarrott, C. and MacDonald, A. (2012). A Review of Extreme Value Threshold Estimation and Uncertainty Quantification. *Revstat Statistical Journal*, 10:33–60.